THE GENERALIZED SPHERICALLY-SYMMETRIC METRIC

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Рассмотрена четырехпараметрическая группа преобразований, содержащая повороты и временной сдвиг, соответствующие сферической симметрии и стационарности метрики пространства-времени. Обнаружено, что существует четверка векторных полей Киллинга, образующих алгебру Ли группы преобразований, и при этом пространственно-подобные векторы не ортогональны время-подобным. Построена метрика, соответствующая алгебре Ли векторов Киллинга. Показано, что метрика нестатична.

Ключевые слова: общая теория относительности, векторы Киллинга, дифференциальная геометрия.

Four parameter group of transformations containing rotations and time translations is considered due to spherical symmetry and stationarity of the space-time metric. It is found that there exists such a quartet of Killing vector fields which constitute the Lie algebra of the transformation group and in which space-like vectors are not orthogonal to the time-like one. The metric corresponding to the Lie algebra of Killing vectors is composed. It is shown that the metric is non-static.

Keywords: general relativity, Killing vectors, differential geometry.

I. Introduction

Stationary spherically-symmetric space-time possessing four Killing vectors [1,2], one of which is time-like and commuting with three others, which are space-like and constitute the Lie algebra so(3) of the group of rotations [3]. Space-times of this kind seemed to be exhausted after the pioneering works of K. Schwartzschild and G. Birkhoff, thus, by 1916 [4-6]. However, there was one missed opportunity because the time-like Killing vector was always assumed to be orthogonal to the space-like ones. This opportunity was explored by A. Taub and C. Misner [7,8] and below we study it thoroughly. Existence of one time-like Killing vector signifies that the space-time can be foliated with identical space-like 3-surfaces. The family of surfaces can be labeled with numerical parameter t which, thus, is a scalar function on the space-time and can be used as time coordinate. Each surface $t=const$ represents space at given shown that infinitesimally-thin rotating Killing vectors which constitute the Lie algebra SO(3):

$$[\vec{L}_1,\vec{L}_2]=-\vec{L}_3,\ \ [\vec{L}_2,\vec{L}_3]=-\vec{L}_1,\ \ [\vec{L}_3,\vec{L}_1]=-\vec{L}_2, \qquad (1)$$

thus, can be foliated with 2-dimensional orbits of the group SO(3) which are spheres in form. The spheres also constitute a linearly-ordered family and can be labeled with parameter r which afterwards serves as the radial coordinate. It is natural to introduce the commonplace coordinates for the spheres which are polar angle θ and azimuthal angle φ, so finally the

space-time is endowed with spherical coordinates $\{t, r, \theta, \varphi\}$ in which the Killing vectors take the form [2]:

$$\vec{L}_1 = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},$$
$$\vec{L}_2 = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \quad (2)$$

$$\vec{L}_3 = \frac{\partial}{\partial\varphi}, \quad \vec{L}_4 = \frac{\partial}{\partial t}. \quad (3)$$

The goal of the present work is to show that there exists such a quartet of Killing vector fields which constitute the Lie algebra of the transformation group and in which space-like vectors are not orthogonal to the time-like one.

II. Generalized killing vectors

Now, let us try to generalize these vectors such a way that commutative relations (1) remain in force while all of them include time-like part:

$$\vec{L}_1 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + a(\theta,\varphi)\frac{\partial}{\partial t},$$
$$\vec{L}_2 = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} + b(\theta,\varphi)\frac{\partial}{\partial t}, \quad (4)$$

$$\vec{L}_3 = \frac{\partial}{\partial\varphi}, \quad \vec{L}_4 = \frac{\partial}{\partial t}. \quad (5)$$

Then spherical symmetry restores in sense that though time shifts are not orthogonal to rotations, but do not distinguish them. Functions a and b are to be found from differential equations which follow from the presumption that Lie algebras (1-3) have same structure:

$$[\vec{L}_4, \vec{L}_a] = 0, \quad [\vec{L}_a, \vec{L}_b] = \varepsilon^c{}_{ab}\vec{L}_c. \quad (6)$$

Substituting the expressions (4, 5) into the commutation relations (1) yields the following differential equations:

$$-\partial_\varphi b = a, \quad \partial_\varphi a = b, \quad (7)$$

from which we conclude that the functions can be represented in the form

$$a = -A(r,\theta)\cos\varphi, \quad b = A(r,\theta)\sin\varphi. \quad (8)$$

All the rest equations reduce to single equation

$$\partial_\theta A - A\cot\theta = 0 \quad (9)$$

whose solution is $A = -l(r)/\sin\theta$. Finally, we have the following extension of the Lie algebra (2, 3):

$$\vec{L}_1 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + \frac{l(r)\cos\varphi}{\sin\theta}\frac{\partial}{\partial t},$$
$$\vec{L}_2 = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - \frac{l(r)\sin\varphi}{\sin\theta}\frac{\partial}{\partial t}, \quad (10)$$

$$\vec{L}_3 = \frac{\partial}{\partial\varphi}, \quad \vec{L}_4 = \frac{\partial}{\partial t}. \quad (11)$$

Explicit form of the function $l(r)$ is to be obtained from the scalar products:

$$\langle\vec{L}_4, \vec{L}_1\rangle = \langle\vec{L}_4, \vec{L}_2\rangle = \langle\vec{L}_4, \vec{L}_3\rangle. \quad (12)$$

III. Generalized spherically-symmetric metric

In fact, we do not know what are "spherical coordinates" in a space-time whose symmetry transformations are generated by the vectors (10, 11). To clear the matter up we try to restore in outlines the form of "spherically-symmetric" metric in which the vectors satisfy the Killing equation [5,10]

$$g_{ij,k}L^k + g_{kj}L^k{}_{,i} + g_{ki}L^k{}_{,j} = 0. \quad (13)$$

Here the indices stand for rotations of coordinates $\{t, r, \theta, \varphi\}$, comma denotes partial derivative on the corresponding coordinate and L^k is the k-component of any of the vectors (10, 11). The subscript r appears only in one term $g_{ki}L^k{}_{,r}$ therefore we have to put $L^k{}_{,r} = 0$, hence, $l(r)=const$.

We expect that the metric to be found is similar to that of spherical coordinates with single non-diagonal component $g_{t\varphi}$, thus,

$$g_{tr} = g_{t\theta} = g_{r\theta} = g_{r\varphi} = g_{\theta\varphi} = 0, \quad (14)$$

and its coefficients but $g_{t\varphi}$ and $g_{\varphi\varphi}$ do not depend on the coordinates θ and φ:

$$g_{tt,\theta} = g_{rr,\theta} = g_{\theta\theta,\theta} = g_{tt,\varphi} = g_{rr,\varphi} = g_{\theta\theta,\varphi} = 0. \quad (15)$$

Finally, we conclude that g_{tt}, g_{rr} and $g_{\theta\theta}$ are functions of single variable r:

$$g_{tt} = f(r), \; g_{rr} = -g(r), \; g_{\theta\theta} = -h(r). \quad (16)$$

In cases of $\vec{L} = \vec{L}_4$ and $\vec{L} = \vec{L}_3$ the equation (13) reduces to:

$$g_{ij,t} = 0, \; g_{ij,\varphi} = 0, \quad (17)$$

therefore all metric coefficients depend only on the coordinates r and θ, so it remains to obtain them as functions of these two variables. The next task is to

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solve the inverse problem on the Killing equation that is to find out the metric under which the vectors (10, 11) satisfy the Killing equation (13). For this end we substitute pairs of indices ($t\theta$), ($t\varphi$), ($\theta\underline{\varphi}$), ($\varphi\varphi$) and derive the corresponding form of metric. These combinations yield two pairs of equations, one for $g_{t\varphi}$:

$$g_{tt}L^{t}{}_{,\theta} + g_{t\varphi}L^{\varphi}{}_{,\theta} = 0, \tag{18}$$

$$g_{t\varphi,\theta}L^{\theta} + g_{tt}L^{t}{}_{,\varphi} + g_{t\varphi}L^{\varphi}{}_{,\varphi} = 0 \tag{19}$$

and one for both $g_{t\varphi}$ and $g_{\varphi\varphi}$:

$$g_{t\varphi}L^{t}{}_{,\theta} + g_{\varphi\varphi}L^{\varphi}{}_{,\theta} + g_{\theta\theta}L^{\theta}{}_{,\varphi} = 0, \tag{20}$$

$$g_{\varphi\varphi,\theta}L^{\theta} + 2g_{t\varphi}L^{t}{}_{,\varphi} + 2g_{\varphi\varphi}L^{\varphi}{}_{,\varphi} = 0. \tag{21}$$

To solve the first pair substitute the vectors (10, 11) that reduces the first equation to the following:

$$\frac{\cos\varphi}{\sin^2\theta}\left(g_{tt}l\cos\theta + g_{t\varphi}\right) = 0. \tag{22}$$

Now we extract the component $g_{t\varphi}$ from this equation and substitute it into (19) to verify that this component satisfies it. The equation reduces to following:

$$f(r)l\sin\theta - \frac{f(r)l}{\sin\theta} + f(r)l\cos\theta\cot\theta = \\ = f(r)l\left(\sin\theta - \frac{1}{\sin\theta} + \cos\theta\cot\theta\right) = 0, \tag{23}$$

thus, the metric coefficient $g_{t\varphi}$ just obtained is the solution of equations (19). Now we substitute it into the (20) and obtain

$$\frac{l^2 f(r)\cos^2\theta}{\sin^2\theta} - \frac{g_{\varphi\varphi}}{\sin^2\theta} + g_{\theta\theta} = 0. \tag{24}$$

Taking into account the equations (16) we obtain the last unknown component of the metric tensor $g_{\varphi\varphi}$:

$$g_{\varphi\varphi} = l^2 f(r)\cos^2\theta - h(r)\sin^2\theta. \tag{25}$$

It remains to substitute this expression into the last equation (21) to make sure that this metric coefficient satisfies both the equations of the second pair. Substituting yields:

$$\partial_\theta[l^2 f(r)\cos^2\theta - h(r)\sin^2\theta]\sin\varphi + \frac{2l^2 f(r)\cos\theta\sin\varphi}{\sin\theta} - 2[l^2 f(r)\cos^2\theta - h(r)\sin^2\theta]\cot\theta\sin\varphi = \\ = -2l^2 f(r)\cos\theta\sin\varphi(\sin\theta - \frac{1}{\sin\theta} + \cos\theta\cot\theta) = 0. \tag{26}$$

Consequently, the metric coefficients $g_{t\varphi}$ and $g_{\varphi\varphi}$ constitute the solution of equations (18-21). Finally, we have generalized spherically-symmetric metric of the form:

$$g_{tt} = f(r),\ \ g_{t\varphi} = -f(r)l\cos\theta,\ \ g_{rr} = -g(r), \\ g_{\theta\theta} = -h(r),\ \ g_{\varphi\varphi} = l^2 f(r)\cos^2\theta - h(r)\sin^2\theta \tag{27}$$

which possesses Killing vectors given by the equation (10, 11).

We suppose that signature of the space-time is (+ -- -- --), so $f(r)$, $g(r)$, and $h(r)$ are non-negative. The metric coefficient $g_{\varphi\varphi}$ is not positively defined, but it is definitely positive under $\theta = \pi/2$. Consider length of the circle in the space t=*const* specified r=*const*, θ =*const*. It is seen that this rate is maximal under $\theta = \pi/2$ and descends out from their curve on the "sphere" r=*const*. Under some value of θ the coefficient $g_{\varphi\varphi}$ vanishes. This zero of the length of the circle specifies the axis. Thus, unlike Schwartzschild space-time, this one has axis under $\theta \neq 0$ $\theta \neq \pi$, but under the zero of $g_{\varphi\varphi}$ found from the equation (26). The equation specifies the range of the coordinate θ. Since $\vec{L}_3 = \partial/\partial\varphi$, its square is negative. Finite rotation generated by space-like Killing vectors can interchange any two of them. Consequently $\vec{L}_1$, $\vec{L}_2$,$\vec{L}_3$ are space-like.

It remains to verify that all scalar products $\langle\vec{L}_a,\vec{L}_4\rangle$ are equal to each other (12). The scalar product $\langle\vec{L}_3,\vec{L}_4\rangle$ is $g_{t\varphi}$ by definition and the scalar product $\langle\vec{L}_2,\vec{L}_4\rangle$ is

$$\langle\vec{L}_2,\vec{L}_4\rangle = \cot\theta\sin\varphi f(r)l\cos\theta - \frac{l\sin\varphi}{\sin\theta}f(r) = \\ = f(r)l\frac{\cos^2\theta - 1}{\sin\theta}\sin\varphi = -f(r)l\sin\theta\sin\varphi \tag{28}$$

that is the same with respect to the corresponding axis. Now consider scalar products of the space-like Killing vectors. The scalar product $\langle\vec{L}_2,\vec{L}_3\rangle$ is:

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$$\langle \vec{L}_2, \vec{L}_3 \rangle = [h(r) + l^2 f(r)] \sin\theta \cos\theta \sin\varphi. \qquad (29)$$

When calculating the product $\langle \vec{L}_1, \vec{L}_2 \rangle$ we omit the common factor $\sin\varphi\cos\varphi$ and all the rest is $-[h(r) + l^2 f(r)]\sin^2\theta$ that yields finally

$$\langle \vec{L}_1, \vec{L}_2 \rangle = -[h(r) + l^2 f(r)] \sin^2\theta \sin\varphi\cos\varphi, \qquad (30)$$

hence, the is same with respect to the corresponding axis metric. Thus, the metric (26) is spherically-symmetric in sense of generalized spherical symmetry discussed above.

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Умумлашган сферик симметрик метрика

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Вақт-фазо метрикаси сферик симметрияси ва стационарлигига мос келувчи, бурилиш ва вақт силжишларини ўз ичига олган, алмаштиришларнинг тўрт параметрли группаси кўриб чиқилди. Алмаштиришлар группасининг Ли алгебрасини ҳосил қилувчи, Киллинг вектор майдонлари тўртлиги мавжудлиги ва фазога ўхшаш векторларнинг вақтга ўхшаш векторларига ортогонал эмаслиги аниқланди. Киллинг векторлари Ли алгебрасига мос келувчи метрика тузилди. Метрика статик эмаслиги кўрсатилди.

Калит сўзлар: умумий нисбийлик назарияси, Киллинг векторлари, дифференциал геометрия.

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